Ordinary differential Equations

UNIT-I : Linear Equation with constant coefficients

Second order homogenous Equations - Initial value problems - Linear dependence and Independence - Wronskian and formula for wronskian - Non homogenous equations of order two

chapter 2 : sec 1 to 6

UNIT-II : Continue

Homogenous and Non-homogenous equation of order n - Initial value problem - Annihilator method to solve non homogenous equation - Algebra of constant coefficient operators.

chapter 2 : sec 7 to 12

UNIT-III : Linear Equations with variable coefficient

Initial value problem - Existence and uniqueness Theorem - Solution to solve a non homogenous equations - wronskian and linear independence - Reduction of the a homogenous equation - homogenous equation with analytic coefficients - The Legendre's equation

chapter 3 : sec 1 to 8

UNIT-IV : Linear Equations with regular singular points

Linear Equation with constant coefficient

Section - 1

Introduction:

A linear differential equation of order n with constant coefficient is of the form

$$a_0 y^n + a_1 y^{n-1} + a_2 y^{n-2} + \dots + a_n y = b(x) \rightarrow ①$$

where $a_0 \neq 0$, $a_1, a_2 \dots a_n$ are complex constant and b is some complex value function on the interval I

$a_0 = 1$, we get

$$y^n + a_1 y^{n-1} + a_2 y^{n-2} + \dots + a_n y = b(x) \rightarrow ②$$

Take $L(y) = y^n + a_1 y^{n-1} + \dots + a_n y \rightarrow ③$

$② \Rightarrow L(y) = b(x) \rightarrow ④$

If $b(x) = 0 \quad \forall \; x \in I$ equation ④ become $L(y) = 0$ and is called homogenous equation.

If $b(x) \neq 0$ equation ② is called non-homogenous equation.

Note:

L is called differential operator. If $\phi(x)$ is a function with derivatives upto order n then

$$L[\phi(x)] = \phi^n(x) + a_1 \phi^{n-1}(x) + \dots + a_n \phi(x)$$

$$\therefore L[\phi] = \phi^n + a_1 \phi^{n-1} + \dots + a_n \phi$$

When ever $L[\phi(x)] = b(x)$ then $\phi(x)$ is called soln of the equation

$$L(y) = b(x)$$

Section - 2

The second order homogenous equation:

Here we are considered the equation

$$L(y) = y'' + a_1 y' + a_2 y = 0 \quad \to (1)$$

Where a_1 and a_2 are constants, We recall that the I^{st} order equation with constant coefficient $y' + ay = 0$ has a soln e^{-ax} the constant 'a' is This soln is the soln of the equation

$$r + a = 0$$

for some appropriate constant r, e^{rx} will be a soln of the equation (1)

$$L[e^{rx}] = [r^2 + a_1 r + a_2]\, e^{rx}$$

and e^{rx} will be a soln of $L(y) = 0$

i.e) $L(e^{rx}) = 0$ if r satisfies

$$r^2 + a_1 r + a_2 = 0$$

We let

$$p(r) = r^2 + a_1 r + a_2$$

and call p the characteristic polynomial

i.e) $L(e^{rx}) = p(r)\, e^{rx}$ $\forall$ r and x.

Theorem: 1

Let a_1, a_2 be a constant and consider the equation $L(y) = y'' + a_1 y' + a_2 y$.

(i) If r_1, r_2 are two distinct roots of the characteristic polynomial $p(r)$ then the function φ_1 and φ_2 are defined by $\varphi_1(x) = e^{r_1 x}$ and $\varphi_2(x) = e^{r_2 x}$
Where $p(r) = \ldots$

Proof:

Consider the equation

(i) $L(y) = y'' + a_1 y' + a_2 y = 0 \rightarrow (1)$

characteristic polynomial is

$p(r) = r^2 + a_1 r + a_2 = 0$

Since r_1, r_2 are roots of $p(r)$ then

$p(r_1) = 0 \ , \ p(r_2) = 0$

$$\left.\begin{aligned} p(r_1) &= r_1^2 + a_1 r_1 + a_2 = 0 \\ p(r_2) &= r_2^2 + a_1 r_2 + a_2 = 0 \end{aligned}\right\} \rightarrow (2)$$

$$\begin{aligned} L[\varphi_1(x)] &= L[e^{r_1 x}] \\ &= (e^{r_1 x})'' + a_1 (e^{r_1 x})' + a_2 e^{r_1 x} \\ &= r_1^2 (e^{r_1 x}) + a_1 r_1 (e^{r_1 x}) + a_2 e^{r_1 x} \\ &= e^{r_1 x}[r_1^2 + a_1 r_1 + a_2] \\ &= e^{r_1 x}(0) \end{aligned}$$

$L[\varphi_1(x)] = 0$

$$\begin{aligned} L[\varphi_2(x)] &= L[e^{r_2 x}] \\ &= (e^{r_2 x})'' + a_1 (e^{r_2 x})' + a_2 e^{r_2 x} \\ &= r_2^2 (e^{r_2 x}) + a_1 r_2 (e^{r_2 x}) + a_2 e^{r_2 x} \\ &= e^{r_2 x}[r_2^2 + a_1 r_2 + a_2] \\ &= e^{r_2 x}(0) \end{aligned}$$

$L[\varphi_2(x)] = 0$

$\therefore \varphi_2(x)$ soln of $L(y) = 0$.

(ii) when $r_1 = r_2$

$p(r_1) = r_1^2 + a_1 r_1 + a_2 = 0 \rightarrow (3)$

$p'(r_1) = 2r_1 + a_1 = 0$

$\varphi(x) = x e^{r_1 x}$

$$\begin{aligned} L[\varphi(x)] &= L[x e^{r_1 x}] \\ &= (x e^{r_1 x})'' + a_1 (x e^{r_1 x})' + a_2 (x e^{r_1 x}) \\ &= (x r_1 e^{r_1 x} + e^{r_1 x})' + a_1 x r_1 e^{r_1 x} + a_1 e^{r_1 x} + a_2 x e^{r_1 x} \end{aligned}$$

$= x r_1^2 e^{r_1 x} + r_1 e^{r_1 x} + r_1 e^{r_1 x} + a_1 x r_1 e^{r_1 x} + a_1 e^{r_1 x} + a_2 x e^{r_1 x}$ (4)

$= x e^{r_1 x}[r_1^2 + a_1 r_1 + a_2] + [2r_1 + a_1] e^{r_1 x}$

$= x e^{r_1 x}(0) + (0) e^{r_1 x}$

$= 0$

$\therefore L[\varphi_2(x)]$ is the soln of $L(y) = 0$

$\varphi_2(x) = x e^{r_1 x}$ is the soln of $L[\varphi(x)]$

Note:

Let $\varphi_1(x), \varphi_2(x)$ be soln of $L(y) = 0$ Then $\varphi = c_1\varphi_1 + c_2\varphi_2$ is also soln of $L(y) = 0$.

Proof:

Since φ_1, φ_2 are soln

$L[y] = y'' + a_1 y' + a_2 y = 0$

$L[\varphi_1] = \varphi_1'' + a_1\varphi_1' + a_2\varphi_1 = 0$

$L[\varphi_2] = \varphi_2'' + a_1\varphi_2' + a_2\varphi_2 = 0$

$L[\varphi] = L[c_1\varphi_1 + c_2\varphi_2]$

$= (c_1\varphi_1 + c_2\varphi_2)'' + a_1(c_1\varphi_1 + c_2\varphi_2)' + a_2(c_1\varphi_1 + c_2\varphi_2)$

$= c_1\varphi_1'' + c_2\varphi_2'' + a_1 c_1\varphi_1' + a_2 c_2\varphi_2' + a_2 c_1\varphi_1 + a_2 c_2\varphi_2$

$= (\varphi_1'' + a_1\varphi_1' + a_2\varphi_1)c_1 + (\varphi_2'' + a_1\varphi_2' + a_2\varphi_2)c_2$

$= (0)c_1 + (0)c_2$

$L(\varphi) = 0$

$\therefore$ Hence $\varphi = c_1\varphi_1 + c_2\varphi_2$ is the soln of $L(y) = 0$.

Section - 3

Initial value problems for second order equations

Consider the equation

$L(y) = y'' + a_1 y' + a_2 y = 0 \rightarrow (1)$

An initial value problem for (1) is the problem of finding a soln φ satisfy

$\varphi(x_0) = \alpha, \quad \varphi'(x_0) = \beta \rightarrow (2)$

where x_0 some real number α and β are two given constant

$\therefore$ The above initial value problem

$L(y) = 0$

$y(x_0) = \alpha$

$y'(x_0) = \beta$

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Theorem: 2 (Existence Theorem)

For any real x_0 and constant α, β there exists a soln of the initial value problem $L(y) = 0$

$y(x_0) = \alpha, \; y'(x_0) = \beta$ on $-\infty < x < \infty$

Proof:

Given x_0 is real, α, β are constant interval is $-\infty < x < \infty$, There are unique constant c_1, c_2 soln

$\varphi = c_1 \varphi_1 + c_2 \varphi_2 \rightarrow (1)$

The soln of initial value problem

$\left.\begin{array}{l} \varphi(x_0) = \alpha \\ \varphi'(x_0) = \beta \end{array}\right\} \rightarrow (2)$

where the soln is φ_1 and φ_2 are given by

$r_1 \neq r_2$

$\left.\begin{array}{l} \varphi_1(x) = e^{r_1 x} \\ \varphi_2(x) = e^{r_2 x} \end{array}\right\} \rightarrow (3)$

$r_1 = r_2$

$\varphi_1(x) = e^{r_1 x}$

$\varphi_2(x) = x e^{r_1 x}$

$$\left.\begin{aligned} c_1\varphi_1(x_0)+c_2\varphi_2(x_0) &= \alpha \\ c_1\varphi_1'(x_0)+c_2\varphi_2'(x_0) &= \beta \end{aligned}\right\} \rightarrow (5)$$

equation (5) will have a unique soln c_1, c_2

if the determinate

$$\Delta = \begin{vmatrix} \varphi_1 & \varphi_2 \\ \varphi_1' & \varphi_2' \end{vmatrix}$$

$$= \varphi_1\varphi_2' - \varphi_2\varphi_1' \neq 0$$

Case (i)

If $r_1 \neq r_2$

$$\varphi_1(x) = e^{r_1 x}, \quad \varphi_1'(x) = r_1 e^{r_1 x}$$

$$\varphi_2(x) = e^{r_2 x}, \quad \varphi_2'(x) = r_2 e^{r_2 x}$$

Now $\Delta = \varphi_1\varphi_2' - \varphi_1'\varphi_2$

$$= e^{r_1 x} r_2 e^{r_2 x} - r_1 e^{r_1 x} e^{r_2 x}$$

$$= e^{r_1 x} e^{r_2 x}(r_2 - r_1)$$

$$= e^{(r_1+r_2)x}(r_2 - r_1)$$

$$\Delta = (r_2 - r_1) e^{(r_1+r_2)x} \neq 0$$

$\therefore \Delta \neq 0$

Case (ii)

If $r_1 = r_2$

$$\varphi_1(x) = e^{r_1 x}, \quad \varphi_1'(x) = r_1 e^{r_1 x}$$

$$\varphi_2(x) = x e^{r_1 x}, \quad \varphi_2'(x) = x r_1 e^{r_1 x} + e^{r_1 x}$$

Now, $\Delta = \varphi_1\varphi_2' - \varphi_1'\varphi_2$

$$= e^{r_1 x}(x r_1 e^{r_1 x} + e^{r_1 x}) - r_1 e^{r_1 x} x e^{r_1 x}$$

$$= x r_1 e^{2 r_1 x} + e^{2 r_1 x} - x r_1 e^{2 r_1 x}$$

$$= e^{2 r_1 x} \neq 0$$

$\Delta \neq 0$

Therefore the determinante condition is satisfied in either case, Thus if c_1, c_2 are the unique constants satisfying the function

$$\varphi = c_1\varphi_1 + c_2\varphi_2$$

Defn:

Let $\varphi(x)$ be any soln of $L(y)=0$ Then the magnitude (size) of $\varphi(x)$ is defined as

$$\|\varphi(x)\| = \sqrt{|\varphi(x)|^2 + |\varphi'(x)|^2}$$

$$\Rightarrow \|\varphi(x)\|^2 = |\varphi(x)|^2 + |\varphi'(x)|^2$$

Also size of z will be measure by

$$K = 1 + |a_1| + |a_2|$$

Result:

If b and c any two constants

$$2|b||c| \le |b|^2 + |c|^2$$

$$0 \le (|b|-|c|)^2$$

$$\le |b|^2 + |c|^2 - 2|b||c|$$

Theorem: 3

Let φ be any soln of $L(y) = y'' + a_1 y' + a_2 y = 0$ on an interval I containing a point x_0 Then for x in I

$$\|\varphi(x_0)\| e^{-K|x-x_0|} \le \|\varphi(x)\| \le \|\varphi(x_0)\| e^{K|x-x_0|}$$

where $\|\varphi(x)\|^2 = |\varphi(x)|^2 + |\varphi'(x)|^2$ and $K = 1 + |a_1| + |a_2|$

Proof:

Given equation

$$L(y) = y'' + a_1 y' + a_2 y = 0 \rightarrow ①$$

and φ is a soln of ①

let $u(x) = \|\varphi(x)\|^2$

$$u(x) = |\varphi(x)|^2 + |\varphi'(x)|^2$$

$$u = |\varphi|^2 + |\varphi'|^2 \qquad |z|^2 = z\bar{z}$$

$$u = \varphi\bar{\varphi} + \varphi'\bar{\varphi}'$$

Differentiate u w.r.t x

$$u' = \varphi'\bar{\varphi} + \varphi\bar{\varphi}' + \varphi''\bar{\varphi}' + \varphi'\bar{\varphi}''$$

$$|u'| \le |\varphi'||\bar{\varphi}| + |\varphi||\bar{\varphi}'| + |\varphi''||\bar{\varphi}'| + |\varphi'||\bar{\varphi}''|$$

$$\le |\varphi'||\varphi| + |\varphi||\varphi'| + |\varphi''||\varphi'| + |\varphi'||\varphi''|$$

$$\le 2|\varphi||\varphi'| + 2|\varphi'||\varphi''| \rightarrow ②$$

Since φ satisfies equation (1) we have

$$L(\varphi)=0$$

$$\varphi''+a_1\varphi'+a_2\varphi=0$$

$$\varphi''=-a_1\varphi'-a_2\varphi$$

$$\Rightarrow \quad |\varphi''|\le |a_1||\varphi'|+|a_2||\varphi| \rightarrow (3)$$

Sub (3) in (2) we get

$$|u'|\le 2|\varphi||\varphi'|+2|\varphi'|\left[|a_1||\varphi'|+|a_2||\varphi|\right]$$

$$\le 2|\varphi||\varphi'|+2|a_1||\varphi'|^2+2|\varphi||\varphi'||a_2|$$

$$|u'|\le 2(1+|a_2|)|\varphi||\varphi'|+2|a_1||\varphi'|^2 \rightarrow (4)$$

W.K.T

$$2|b||c|\le |b^2|+|c^2| \rightarrow (5)$$

using (5) in (4)

$$|u'|\le(1+|a_2|)\left[|\varphi|^2+|\varphi'|^2\right]+2|a_1||\varphi'|^2$$

$$\le(1+|a_2|)|\varphi|^2+(1+|a_2|)|\varphi'|^2+2|a_1||\varphi'|^2$$

$$\le(1+|a_2|)|\varphi|^2+(1+2|a_1|+|a_2|)|\varphi'|^2 \rightarrow (6)$$

This inequality

$$\Rightarrow e^{-2Ku}(u'-2Ku) \le 0$$

$$(e^{-2Ku} u)' \le 0$$

If x is $\ge x_0$ we integrate this equation with respect x, x_0 to x

$$\int_{x_0}^{x} (e^{-2Ku} u)' \le 0$$

$$[e^{-2Ku} u]_{x_0}^{x} \le 0$$

$$e^{-2Kx} u(x) - e^{-2Kx_0} u(x_0) \le 0$$

$$e^{-2Kx} u(x) \le e^{-2Kx_0} u(x_0)$$

$$u(x) \le e^{2K(x-x_0)} u(x_0)$$

$$\Rightarrow \|\varphi(x)\|^2 \le e^{2K(x-x_0)} \|\varphi(x_0)\|^2$$

$$\Rightarrow \|\varphi(x)\| \le e^{K|x-x_0|} \|\varphi(x_0)\| \rightarrow (7)$$

Similarly consider the left inequality of left limit $x < x_0$

$$\|\varphi(x_0)\| e^{-K|x-x_0|} \le \|\varphi(x)\| \rightarrow (8)$$

Combine (7) and (8)

$$\|\varphi(x_0)\| e^{-K|x-x_0|} \le \|\varphi(x)\| \le \|\varphi(x_0)\| e^{K|x-x_0|}$$

Theorem: 4 (uniqueness Theorem)

Let α, β are any two constant and let x_0 be any real number on any interval I containing x_0 atmost one soln φ of the I.V.P $L(y)=0$, $y(x_0)=\alpha$, $y'(x_0)=\beta$

Proof:

Let φ and ψ, we two soln of the given problems

Let $\theta = \varphi - \psi \rightarrow (1)$

Then $L(\theta) = L(\varphi) - L(\psi)$

$= 0 - 0$

$L(\theta) = 0$

by ①

$$\theta(x_0) = \phi(x_0) - \psi(x_0)$$

$$= \alpha - \alpha$$

$$\theta(x_0) = 0$$

Similarly $\theta'(x_0) = \phi'(x_0) - \psi'(x_0)$

$$= \beta - \beta$$

$$\theta'(x_0) = 0$$

$$\|\theta(x_0)\|^2 = |\theta(x_0)|^2 + |\theta'(x_0)|^2$$

$$= 0 + 0$$

$$\|\theta(x_0)\|^2 = 0$$

using The inequality of previous Theorem

$$\|\phi(x_0)\| e^{-K|x-x_0|} \le \|\phi(x)\| \le \|\phi(x_0)\| e^{K|x-x_0|} \rightarrow ②$$

from the function of θ

$\|\theta(x_0)\| = 0 \quad \forall\ x \text{ in } I$

$\Rightarrow \theta(x_0) = 0 \quad \forall\ x \text{ in } I$

$\Rightarrow \phi = \psi$

Hence atmost one soln ϕ of the initial value problems

$y(x_0) = \alpha, \quad y'(x_0) = \beta, \quad L(y) = 0$

Theorem: 5

Let ϕ_1, ϕ_2 be two soln of $L(y)=0$ given by $\phi_1(x) = e^{r_1 x}$, $\phi_2(x) = e^{r_2 x}$ in case $r_1 \neq r_2$ (distinct) and $r_1 = r_2$ (Repeated). If c_1, c_2 are any two constants The function $\phi = c_1\phi_1 + c_2\phi_2$ is a soln of $L(y)=0$ on $-\infty < x < \infty$.

Conversely if ϕ is any soln of $L(y)=0$ on $-\infty < x < \infty$ There are unique constant c_1, c_2 such that $\phi = c_1\phi_1 + c_2\phi_2$.

Proof:

First Part

Let ϕ_1, ϕ_2 be two solns of $L(y)=0$ That is

$\phi_1(x) = e^{r_1 x}$, $\phi_2(x) = e^{r_2 x}$

If ϕ_1, ϕ_2

$L(\phi_1)=0$, $L(\phi_2)=0$

Assume that ϕ is any soln of $L(y)=0$, on $-\infty<x<\infty$

There are unique constant c_1, c_2 suchthat

$\phi = c_1\phi_1 + c_2\phi_2$

We prove that

$L[\phi] = L[c_1\phi_1 + c_2\phi_2] = 0$

$L[y] = y'' + a_1 y' + a_2 y = 0$

$L(\phi) = (c_1\phi_1 + c_2\phi_2)'' + a_1(c_1\phi_1 + c_2\phi_2)' + a_2(c_1\phi_1 + c_2\phi_2)$

$= c_1\phi_1'' + c_2\phi_2'' + a_1 c_1\phi_1' + a_1 c_2\phi_2' + a_2 c_1\phi_1 + a_2 c_2\phi_2$

$= c_1(\phi_1'' + a_1\phi_1' + a_2\phi_1) + c_2(\phi_2'' + a_1\phi_2' + a_2\phi_2)$

$= c_1 L(\phi_1) + c_2 L(\phi_2)$

$= c_1(0) + c_2(0)$

$L(\phi) = 0$

$\therefore \phi$ is the soln of $L(y)=0$

$\Rightarrow$ there exists constant c_1, c_2 suchthat

$\phi = c_1\phi_1 + c_2\phi_2$

Converse Part:

Let ϕ be the soln x_0 is real $\phi(x_0)=\alpha$, $\phi'(x_0)=\beta$,

By Existence theorem there exists a soln ψ of

$L(y)=0$,

$\psi(x_0)=\alpha$

$\psi'(x_0)=\beta$ suchthat

$\psi = c_1\psi_1 + c_2\psi_2$

where c_1, c_2 are unique constants determinates by α, β

by uniqueness theorem

$\phi = \psi$

Hence the theorem.

Section- 4

Linear dependent and independent

Defn: linearly Dependent

Two functions φ_1, φ_2 on an interval I are said to be linearly dependent on I if $\exists$ two constants not both zero such that

$$c_1\varphi_1(x) + c_2\varphi_2(x) = 0 \quad \forall\ x \text{ in } I$$

Defn: Linearly Independent

The functions φ_1 and φ_2 are said to be linearly independent on I. If the constant c_1 and c_2 are such that

$$c_1\varphi_1(x) + c_2\varphi_2(x) = 0 \quad \forall\ x \text{ in } I$$

$$\Rightarrow c_1 = c_2 = 0$$

(or)

Two functions φ_1 and φ_2 are said to be linearly independent on I. If there not linearly dependent on I.

Wronskian of two functions:

Wronskian of two functions φ_1 and φ_2 defined on I is denotes $W(\varphi_1, \varphi_2)$ is given as

$$W(\varphi_1, \varphi_2) = \begin{vmatrix} \varphi_1 & \varphi_2 \\ \varphi_1' & \varphi_2' \end{vmatrix}$$

$$= \varphi_1\varphi_2' - \varphi_2\varphi_1'$$

Example:

Let $\varphi_1(x) = \sin x$, $\varphi_2(x) = \cos x$

$\varphi_1'(x) = \cos x$, $\varphi_2'(x) = -\sin x$

$$W(\varphi_1, \varphi_2)(x) = \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix}$$

$$= -\sin^2 x - \cos^2 x$$

$$= -(\sin^2 x + \cos^2 x)$$

$$W(\varphi_1, \varphi_2)(x) = -1$$

Theorem: 6

Two solns φ_1, φ_2 of $L(y)=0$ are linearly independent on Interval I, iff $W(\varphi_1, \varphi_2)(x) \neq 0$ $\forall$ x in I.

Proof:

Let us suppose $W(\varphi_1, \varphi_2)(x) \neq 0$ $\forall$ $x \in I$ and

Let c_1, c_2 be constants $\ni$

$$c_1\varphi_1(x) + c_2\varphi_2(x) = 0 \rightarrow (1) \quad \forall x \in I$$

Also by differential (1) w.r.t x

$$c_1\varphi_1'(x) + c_2\varphi_2'(x) = 0 \rightarrow (2) \quad \forall x \in I$$

for a fixed x, The equation (1) and (2) are linearly homogenous equations satisfied by c_1, c_2

The determinant of the coefficient is $W(\varphi_1, \varphi_2)(x)$ which is not zero.

$\therefore$ $c_1 = 0$, $c_2 = 0$ is only soln of (1) and (2)

$\therefore$ φ_1, φ_2 are linearly independent on I.

Conversely

Let us assume φ_1, φ_2 are linearly independent on I

T.P: $W(\varphi_1, \varphi_2)(x) \neq 0$ $\forall$ $x \in I$

There exists $W(\varphi_1, \varphi_2)(x_0) = 0$

Take the equations

$$c_1\varphi_1(x_0) + c_2\varphi_2(x_0) = 0$$
$$c_1\varphi_1'(x_0) + c_2\varphi_2'(x_0) = 0$$

Where c_1 and c_2 are constant

There are linearly homogenous equations.

Here the determinant of the constant

ie) $W(\varphi_1, \varphi_2)(x_0) = 0$

$\therefore$ Atleast on of the constants c_1, c_2 is not zero

For the constants c_1, c_2 We have

$$c_1\varphi_1(x) + c_2\varphi_2(x) = 0 \quad \forall x \in I$$

~~one~~ when c_1, c_2 is not zero.

Let c_1, c_2 such a soln consider the function

$\psi = c_1 \varphi_1 + c_2 \varphi_2$

Now, $L(\psi) = 0$

$\psi(x_0) = 0, \quad \psi'(x_0) = 0$

$\Rightarrow \varphi_1, \varphi_2$ are linearly independent on I

The contradicts the statement of the theorem

$W(\varphi_1, \varphi_2)(x) \neq 0 \quad \forall x \in I$

Hence the theorem.

Theorem: 7

Let φ_1, φ_2 two solns of $L(y) = 0$ on an interval I and let x_0 be any point in I Then φ_1, φ_2 are linearly independent on I iff $W(\varphi_1, \varphi_2)(x_0) \neq 0$

Proof:

Let φ_1, φ_2 be linearly independent on I

by above theorem

$W(\varphi_1, \varphi_2)(x) \neq 0 \quad \forall x \in I$

In particular

$W(\varphi_1, \varphi_2)(x_0) \neq 0$

conversely

let $W(\varphi_1, \varphi_2)(x_0) \neq 0$

claim:

φ_1, φ_2 are linearly independent on I.

consider the equations

$$c_1 \varphi_1(x_0) + c_2 \varphi_2(x_0) = 0$$
$$c_1 \varphi_1'(x_0) + c_2 \varphi_2'(x_0) = 0 \quad \forall x_0 \in I$$

where c_1, c_2 are constants.

Since the determinant of above linearly homogenous equation is

$$W(\varphi_1, \varphi_2)(x_0) = \begin{vmatrix} \varphi_1(x_0) & \varphi_2(x_0) \\ \varphi_1'(x_0) & \varphi_2'(x_0) \end{vmatrix} = \varphi_1(x_0)\varphi_2'(x_0) - \varphi_2(x_0)\varphi_1'(x_0)$$

we obtained $c_1 \varphi_1(x) + c_2 \varphi_2(x) = 0 \quad \forall x \in I$

and $c_1 = c_2 = 0$

Thus φ_1, φ_2 are linearly independent on I.

Theorem: 8

Let φ_1, φ_2 are any two L.I soln of $L(y)=0$ on an interval I. Then every soln φ of $L(y)=0$ can be written uniquely as $\varphi = c_1\varphi_1 + c_2\varphi_2$ where c_1, c_2 are constant.

Proof:

Let x_0 be a point in I

Given that φ_1, φ_2 are L.I on I

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$$W(\varphi_1, \varphi_2)(x_0) \neq 0$$

Let φ be any soln with $\varphi(x_0)=\alpha$, $\varphi'(x_0)=\beta$

Consider two equations

$$c_1\varphi_1(x_0) + c_2\varphi_2(x_0) = \alpha = \varphi(x_0) \rightarrow (1)$$

$$c_1\varphi_1'(x_0) + c_2\varphi_2'(x_0) = \beta = \varphi'(x_0) \rightarrow (2)$$

where c_1, c_2 are constants satisfying (1) and (2)

Let c_1, c_2 be these constants Then the functions

$\psi = c_1\varphi_1 + c_2\varphi_2$ is such that

$$\psi(x_0) = c_1\varphi_1(x_0) + c_2\varphi_2(x_0)$$

$$= \alpha$$

$$\psi(x_0) = \varphi(x_0)$$

$$\psi'(x_0) = c_1\varphi_1'(x_0) + c_2\varphi_2'(x_0)$$

$$= \beta$$

$$\psi'(x_0) = \varphi'(x_0)$$

$$L(\psi) = L(c_1\varphi_1 + c_2\varphi_2) = 0$$

$$\psi = \varphi$$

$\therefore \psi$ is a soln of φ is $L(y)=0$

uniqueness Theorem

$$\varphi = c_1\varphi_1 + c_2\varphi_2$$

Hence proved.

Section-5

A formula for the wronskian

Theorem: 9

If φ_1, φ_2 are two solns of $L(y)=0$ on interval I containing at point x_0 Then

$$w(\varphi_1,\varphi_2)(x) = e^{-a_1(x-x_0)}\, w(\varphi_1,\varphi_2)(x_0)$$

Proof:

Since φ_1, φ_2 are two soln of

$$L(y) = y'' + a_1 y' + a_2 y = 0 \quad \rightarrow (1)$$

We have

$$L(\varphi_1) = \varphi_1'' + a_1\varphi_1' + a_2\varphi_1 = 0 \quad \rightarrow (2)$$

$$L(\varphi_2) = \varphi_2'' + a_1\varphi_2' + a_2\varphi_2 = 0 \quad \rightarrow (3)$$

$(2) \times \varphi_2 \Rightarrow \varphi_2\varphi_1'' + a_1\varphi_1'\varphi_2 + a_2\varphi_1\varphi_2 = 0$

$(3) \times \varphi_1 \Rightarrow \varphi_1\varphi_2'' + a_1\varphi_2'\varphi_1 + \cancel{a_2\varphi_1\varphi_2} = 0$

$$(\varphi_1\varphi_2'' - \varphi_2\varphi_1'') + a_1(\varphi_1\varphi_2' - \varphi_2\varphi_1') = 0 \quad \rightarrow (4)$$

If $w = w(\varphi_1,\varphi_2)$

$$= \begin{vmatrix} \varphi_1 & \varphi_2 \\ \varphi_1' & \varphi_2' \end{vmatrix}$$

$$w = \varphi_1\varphi_2' - \varphi_1'\varphi_2 \quad \rightarrow (5)$$

$$w' = \varphi_1\varphi_2'' + \varphi_1'\varphi_2' - \varphi_1'\varphi_2' - \varphi_2\varphi_1''$$

$$w' = \varphi_1\varphi_2'' - \varphi_2\varphi_1'' \quad \rightarrow (6)$$

sub (6) and (5) in (4)

$$w' + a_1 w = 0$$

$$\frac{dw}{dx} + a_1 w = 0$$

$$\frac{dw}{dx} = -a_1 w$$

$$\frac{dw}{w} = -a_1\,dx$$

$$\int \frac{dw}{w} = -a_1 \int dx$$

$$\log w = -a_1 x + \log c$$

$$\log w - \log c = -a_1 x$$

$\log\left(\frac{\omega}{c}\right) = -a_1 x$

$\omega/c = e^{-a_1 x}$

$\omega = c\, e^{-a_1 x} \rightarrow (7)$

put $x = x_0$

$\omega(x_0) = c\, e^{-a_1 x_0}$

$c = \omega(x_0)\, e^{a_1 x_0} \rightarrow (8)$

Sub (8) in (7)

$\omega = \omega(x_0)\, e^{a_1 x_0}\, e^{-a_1 x}$

$\omega = \omega(x_0)\, e^{-a_1(x - x_0)}$

$\Rightarrow \omega(\varphi_1, \varphi_2)(x) = e^{-a_1(x-x_0)}\, \omega(\varphi_1, \varphi_2)(x_0)$

Section - 6

The Non-homogenous equations of order two

Theorem: 10

Let b be continuous on an interval I. Every soln ψ of $L(y) = b(x)$ on I can be written as $\psi = \psi_p + c_1\varphi_1 + c_2\varphi_2$. Where ψ_p is a particular soln φ_1, φ_2 are two L.I soln of $L(y) = 0$ and c_1, c_2 are constants. A particular soln ψ_p

i.e) Given That $\psi_p(x) = \int_{x_0}^{x} \frac{\varphi_1(t)\varphi_2(x) - \varphi_1(x)\varphi_2(t)}{\omega(\varphi_1, \varphi_2)(t)}\, b(t)\, dt$

conversely every such ψ is a soln of $L(y) = b(x)$.

Proof:

Let us consider the homogenous equation

$L(y) = y'' + a_1 y' + a_2 y = 0 \rightarrow (1)$

Given φ_1, φ_2 are two linear independent of $L(y) = 0$

i.e) soln is $c_1\varphi_1 + c_2\varphi_2 \rightarrow (2)$

where c_1, c_2 are constants

Let us choose

$$\psi_p(x) = u_1(x)\,\varphi_1(x) + u_2(x)\,\varphi_2(x) \rightarrow (3)$$

be a particular soln of

$$L(y) = b(x) \rightarrow (4)$$

$$L(\psi_p) = b(x)$$

ie) $L[\psi_p] = \psi_p'' + a_1\psi_p' + a_2\psi_p = b(x) \rightarrow (5)$

$$\psi_p = u_1\varphi_1 + u_2\varphi_2$$

$$\psi_p' = u_1\varphi_1' + u_1'\varphi_1 + u_2\varphi_2' + u_2'\varphi_2 \rightarrow (6)$$

$$\psi_p'' = u_1\varphi_1'' + \varphi_1'u_1' + u_1'\varphi_1' + u_1''\varphi_1 + u_2\varphi_2'' + u_2'\varphi_2' + u_2'\varphi_2' + u_2''\varphi_2$$

$$\psi_p'' = u_1\varphi_1'' + 2u_1'\varphi_1' + 2u_2'\varphi_2' + u_1''\varphi_1 + u_2\varphi_2'' + u_2''\varphi_2 \rightarrow (7)$$

sub (3), (6), (7) in (5)

$$\left.\begin{aligned} &(u_1\varphi_1'' + 2u_1'\varphi_1' + 2u_2'\varphi_2' + u_1''\varphi_1 + u_2\varphi_2'' + u_2''\varphi_2) + \\ &a_1[u_1\varphi_1' + u_1'\varphi_1 + u_2\varphi_2' + u_2'\varphi_2] + \\ &a_2[u_1\varphi_1 + u_2\varphi_2] \end{aligned}\right\} = b(x)$$

$$\left.\begin{aligned} &u_1[\varphi_1'' + a_1\varphi_1' + a_2\varphi_1] + u_2[\varphi_2'' + a_1\varphi_2' + a_2\varphi_2] \\ &+ 2[u_1'\varphi_1' + u_2'\varphi_2'] + [u_1''\varphi_1 + u_2''\varphi_2] + \\ &\qquad a_1[u_1'\varphi_1 + u_2'\varphi_2] \end{aligned}\right\} = b(x)$$

$$\left.\begin{aligned} &2[u_1'\varphi_1' + u_2'\varphi_2'] + [u_1''\varphi_1 + u_2''\varphi_2] + \\ &\qquad a_1[u_1'\varphi_1 + u_2'\varphi_2] \end{aligned}\right\} = b \longrightarrow (*)$$

assume

$$u_1'\varphi_1 + u_2'\varphi_2 = 0 \rightarrow (8)$$

$$[u_1'\varphi_1 + u_2'\varphi_2]' = 0 \rightarrow (9)$$

$$u_1''\varphi_1 + u_1'\varphi_1' + u_2''\varphi_2 + u_2'\varphi_2' = 0$$

$$(u_1''\varphi_1 + u_2''\varphi_2) + (u_1'\varphi_1' + u_2'\varphi_2') = 0 \rightarrow (10)$$

sub (8), (9), (10) in (*)

$$u_1'\varphi_1' + u_2'\varphi_2' = b \rightarrow (11)$$

equation ⑧ and ⑪ are two linear equation for u_1', u_2' with the determinate which is $w(\varphi_1', \varphi_2')$ This w is never zero on I. Because φ_1, φ_2 are linearly independent

$\therefore$ There exists unique soln of u_1', u_2'

$$w(\varphi_1, \varphi_2) = \begin{vmatrix} \varphi_1 & \varphi_2 \\ \varphi_1' & \varphi_2' \end{vmatrix}$$

$$= \varphi_1\varphi_2' - \varphi_1'\varphi_2 \rightarrow ⑫$$

$$⑧ \times \varphi_1' \Rightarrow \cancel{u_1'\varphi_1'\varphi_1} + u_2'\varphi_1'\varphi_2 = 0$$

$$⑪ \times \varphi_1 \Rightarrow \underset{(-)}{\cancel{u_1'\varphi_1'\varphi_1}} + \underset{(-)}{u_2'\varphi_1\varphi_2'} = \underset{(-)}{b\varphi_1}$$

$$u_2'(\varphi_1'\varphi_2 - \varphi_1\varphi_2') = -b\varphi_1$$

$$-u_2'(\varphi_1\varphi_2' - \varphi_1'\varphi_2) = -b\varphi_1$$

$$u_2' = \frac{b\varphi_1}{(\varphi_1\varphi_2' - \varphi_1'\varphi_2)}$$

$$u_2' = \frac{b\varphi_1}{w(\varphi_1, \varphi_2)}$$

Similarly

$$u_1' = \frac{-\varphi_2 b}{w(\varphi_1, \varphi_2)}$$

If x_0 is in I, we get

$$u_1 = -\int_{x_0}^{x} \frac{\varphi_2(t)\, b(t)}{w(\varphi_1, \varphi_2)(t)}\, dt$$

$$u_2' = \int_{x_0}^{x} \frac{\varphi_1(t)\, b(t)}{w(\varphi_1, \varphi_2)(t)}\, dt$$

The soln $\psi_p = u_1\varphi_1 + u_2\varphi_2$ Then takes the form

$$\psi_p(x) = \int_{x_0}^{x} \left[\frac{\varphi_1(t)\,\varphi_2(x) - \varphi_2(t)\,\varphi_1(x)}{w(\varphi_1, \varphi_2)(t)} \right] b(t)\, dt$$

Converse part

W.K.T

$$L(y) = y'' + a_1 y' + a_2 y = b(x)$$